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One-dimensional constitutive relation for shape-memory alloy-reinforced composite lamina

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Based on the one-dimensional thermo-mechanical constitutive relation of a shape-memory alloy (SMA) in which the dependence of the elastic modulus of SMA upon the martensite fraction is considered, a one-dimensional constitutive relation for SMA-reinforced composite lamina has been developed. The stress–strain relation under constant temperature, the free recovery and the restrained recovery under variable temperature, have been analysed for the SMA-reinforced lamina.

1. Introduction

In 1965, Buehler and Wiley of the US Naval Ordnance Laboratory developed a series of alloys which possess a unique mechanical property, i.e. the shape-memory effect (SME). These alloys contain 53%–57 wt % nickel, known as 55-Nitinol. The shape-memory effect occurs when a material is stressed to induce martensitic transformation at low temperature and is plastically deformed, but upon removal of the stress, it regains the original shape on heating. Many materials, such as the copper alloy systems, alloys of Au–Cd, Ni–Al, etc., are now known to exhibit the shape memory effect. In the literature, those materials such as the shape-memory alloy, piezo-electric polymers and ceramics, electro-rheological fluids and optical fibres, which have the unique abililty to change their physical properties (stiffness, damping, shape, viscosity, etc.) due to external stimuli, are referred to as "smart materials". However, research on SMA-reinforced composites, pioneered by Rogers and Robertshaw [1] at VPI&SU, only began recently. One of the prospective applications of SMA-reinforced composites is the active vibration control of large flexible aerospace and space structures. It is suggested that upon heating of the embedded SMA wires, the stiffness as well as the strain energy of the structure can be actively tuned, and thus the dynamic response of the structure can be actively controlled.

In the present study, a one-dimensional constitutive relation of the composite lamina with embedded SMA wires has been derived. In the formulation, the dependence of the elastic modulus of the SMA wire on the variation of martensite fraction is assumed.

2. One-dimensional constitutive relation of SMA-reinforced lamina

Tanaka [2] has demonstrated that the thermo-mechanical behaviour of SMA during the course of stress-induced martensitic transformation or the reverse transformation, can be described in the following uniaxial form

$$\dot{\sigma}_a = D\dot{\varepsilon}_a + \theta\dot{T} + \Omega\dot{\xi} \tag{1}$$

where σ_a and ε_a are the stress and the strain of the embedded SMA. In Tanaka's original work, σ_a and ε_a are referred to as the components of the Piola–Kirchhoff stress tensor and the Green strain tensor, respectively. In the present study only a small strain was assumed because the fibrous composites usually work in the range of small strain. D, θ and Ω are elastic modulus, thermo-coefficient and phase-transformation coefficient of SMA, respectively. ξ is the martensite fraction ($0 < \xi < 1$), T is the temperature. The elastic modulus D is in general a function of temperature, T, and martensite fraction, ξ. Tanaka [2] and Liang and Rogers [3] assumed D to be a constant throughout the process. However, the elastic modulus of SMA in the martensite phase is often much less than that in the austenite phase. The elastic modulus of 55-Nitinol in the austenite phase ($D_A = 75.9$ GPa), for example, is more than three times as high as that in the martensite phase ($D_M = 24.1$ GPa). When considering the effect of changeable elastic modulus, D, of embedded SMA on the stiffness of the SMA-reinforced composite structure, it is assumed that D is a function of ξ in this study.

Equation 1 can be written in the following differential form

$$D(\xi)\,\mathrm{d}\varepsilon_a = \mathrm{d}\sigma_a - \Omega\mathrm{d}\xi - \theta\,\mathrm{d}T \quad (2)$$

The elastic modulus $D(\xi)$ is assumed to be in the form

$$D(\xi) = \frac{D_A - D_M}{2}(\cos\xi\pi + 1) + D_M \quad (3)$$

The matrix in which the SMA wires are embedded, is a conventional fibre/resin composite. In the following, the stress and the strain of the matrix in the SMA wire's direction are denoted σ_m and ε_m, the elastic modulus and the thermal expansion coefficient of the matrix in the same direction are denoted E_m and α_m, the volume fraction of the SMA wires and the matrix are denoted as V_a and V_m, and the stress and the strain of the SMA-reinforced composite lamina are denoted as σ and ε, respectively. Liang and Roger [3] used the cosine function to describe the dependence of ε on T and σ_a of SMA. During the course of the phase transformation from martensite to austenite, ε is given by

$$\varepsilon = \tfrac{1}{2}\xi_M(\cos[a_A(T - A_s) - b_A\sigma_a] + 1) \quad (4)$$

and during the course of martensitic transformation

$$\varepsilon = \tfrac{1}{2}(1 - \xi_A)\cos[a_M(T - M_f) - b_M\sigma_a] + \frac{1 + \xi_A}{2} \quad (5)$$

where the material constants are

$$a_A = \frac{\pi}{A_f - A_s} \quad (6a)$$

$$a_M = \frac{\pi}{M_s - M_f} \quad (6b)$$

$$b_A = \frac{a_A}{C_A} \quad (6c)$$

$$b_M = \frac{a_M}{C_M} \quad (6d)$$

where A_s, A_f, M_s and M_f are the austenite and martensite start and finish temperatures of the SMA under stress-free conditions, respectively. C_A and C_M are SMA material constants related to stress-induced phase transformation. ξ_M and ξ_A are initial martensite fractions when the M $\rightarrow$ A or the A $\rightarrow$ M transformation starts from a state which has mixed austenite and martensite phases. Fig. 1 shows the dependence of martensite fraction, ξ, on temperature, T, under stress-free conditions. The upper curve depicts the change of ξ in the course of M $\rightarrow$ A transformation when heating, while the lower one depicts the A $\rightarrow$ M transformation when cooling.

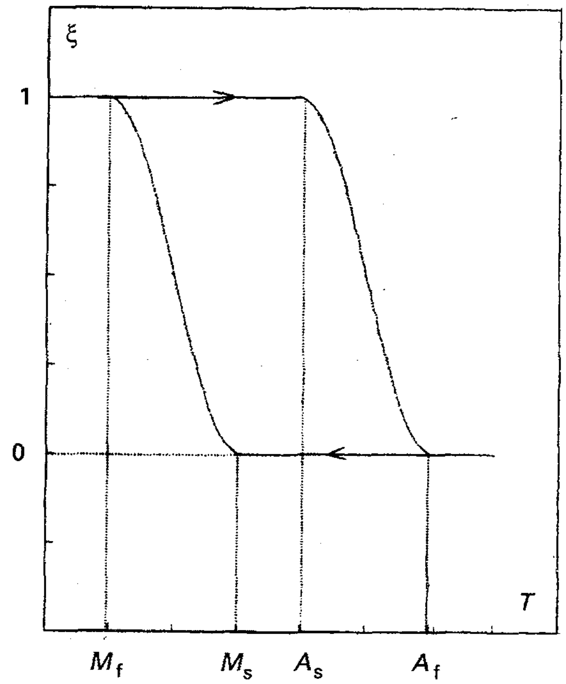

Figure 1 Relation of ξ with temperature, T.

As shown in Fig. 2, when the SMA-reinforced composite lamina is uniaxially stressed, the strain of the SMA wires and the matrix are assumed to be the same and to equal the laminar strain

$$\mathrm{d}\varepsilon_m = \mathrm{d}\varepsilon_a = \mathrm{d}\varepsilon \quad (7)$$

From the equilibrium considerations, we have

$$\mathrm{d}\sigma = V_a\mathrm{d}\sigma_a + V_m\mathrm{d}\sigma_m \quad (8)$$

From the one-dimensional thermo-elastic relation of the matrix, we have

$$\mathrm{d}\varepsilon_m = \frac{\mathrm{d}\sigma_m}{E_m} + \alpha_m\mathrm{d}T \quad (9)$$

Eliminating $\mathrm{d}\sigma_m$ from Equations 8 and 9 leads to

$$\mathrm{d}\varepsilon_m = \frac{1}{E_m V_m}(\mathrm{d}\sigma - V_a\mathrm{d}\sigma_a) + \alpha_m\mathrm{d}T \quad (10)$$

Using the relation

$$\mathrm{d}\xi = \frac{\partial\xi}{\partial\sigma_a}\mathrm{d}\sigma_a + \frac{\partial\xi}{\partial T}\mathrm{d}T$$

Equation 2 can be rewritten as

$$\mathrm{d}\varepsilon_a = \frac{1}{D(\xi)}\left[\left(1 - \Omega\frac{\partial\xi}{\partial\sigma_a}\right)\mathrm{d}\sigma_a - \left(\theta + \Omega\frac{\partial\xi}{\partial T}\right)\mathrm{d}T\right] \quad (11)$$

Eliminating σ_a from Equations 10 and 11 and using Equation 7, the differential relation of σ, ε and T is given by

$$\left[D(\xi) + \frac{V_m E_m}{V_a}\left(1 - \Omega\frac{\partial\xi}{\partial\sigma_a}\right)\right]\mathrm{d}\varepsilon = \left[\left(1 - \Omega\frac{\partial\xi}{\partial\sigma_a}\right)\Big/V_a\right]\mathrm{d}\sigma + \left\{\left[\left(1 - \Omega\frac{\partial\xi}{\partial\sigma_a}\right)\Big/V_a\right]E_m\alpha_m V_m - \theta - \Omega\frac{\partial\xi}{\partial T}\right\}\mathrm{d}T \quad (12)$$

TABLE I Material properties of SMA

D_A(MPa)	D_M(MPa)	Ω	θ(MPa °C^{-1})	M_f(°C)	M_s(°C)	A_s(°C)	A_f(°C)	C_M(MPa °C^{-1})	C_A(MPa °C^{-1})
75.9×10^3	24.1×10^3	−700	0.1	−20	16	38	74	2.5	2.5

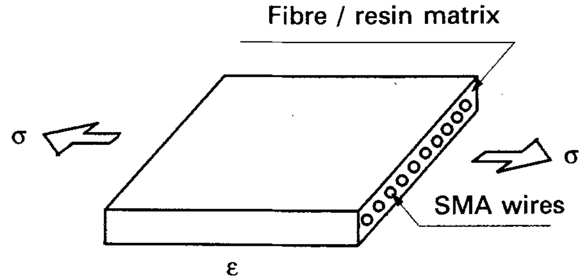

Figure 2 An SMA-reinforced lamina.

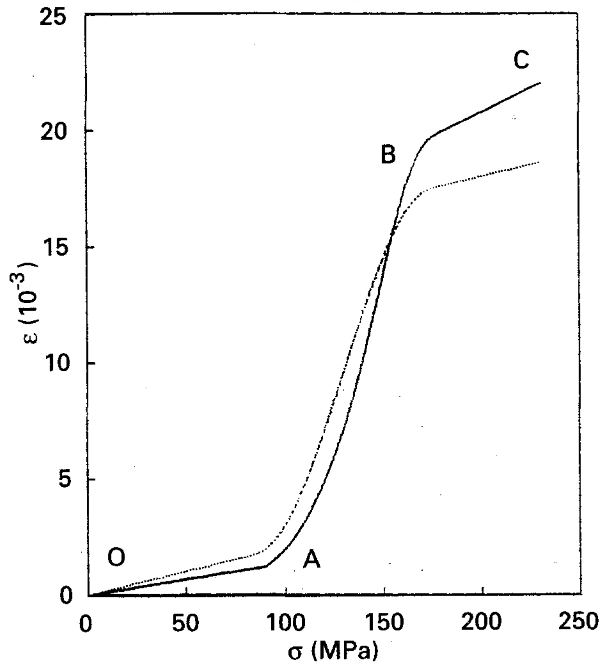

Figure 3 The stress–strain relation of SMA wire ($T_0 = 50$ °C).

Equation 12 is the differential constitutive relation of stress, σ, strain, ε, and temperature, T of the SMA-reinforced composite lamina. In Equation 12 both $\partial\xi/\partial\sigma_a$ and $\partial\xi/\partial T$ are functions of σ_a and T, and σ_a is related to σ and ε in the form

$$\sigma_a = \frac{1}{V_a}\{\sigma - V_m E_m[\varepsilon - \alpha_m(T - T_0)]\} \quad (13)$$

where T_0 is the initial temperature. Finally, Equation 12 contains only variables σ, ε and T and their differentials. In a process under constant temperature, Equation 12 becomes an ordinal differential equation between σ and ε because $dT = 0$. The relation of σ and ε can be obtained by numerical integration such as the Runga–Kutta method. In a process free from external stress (i.e. $\sigma = 0$), Equation 12 gives the relation between strain, ε, and temperature, T. The curve $\varepsilon(T)$ can be obtained by integrating Equation 12. The condition $d\varepsilon = 0$ means that two ends of the lamina are fully restrained. In this case, Equation 12 gives the relation of σ and T with $\varepsilon = 0$. Using Equations 4 and 5, the partial differential of ξ with respect to σ_a and T is given by

TABLE II Material properties of the matrix

Matrix	E_{11}(GPa)	E_{22}(GPa)	α_1(10^{-6} °C^{-1})	α_2(10^{-6} °C^{-1})
Graphite/ epoxy	146	10.8	−1.1	43.05
Glass/epoxy	39.3	8.3	6.6	19.7

$$\frac{\partial\xi}{\partial\sigma_a} = \frac{\xi_M}{2} b_A \sin[a_A(T - A_s) - b_A\sigma_a] \quad (14a)$$

$$\frac{\partial\xi}{\partial T} = -\frac{\xi_M}{2} a_A \sin[a_A(T - A_s) - b_A\sigma_a] \quad (14b)$$

For the transformation from martensite to austenite, and

$$\frac{\partial\xi}{\partial\sigma_a} = \frac{1 - \xi_A}{2} b_M \sin[a_M(T - M_f) - b_M\sigma_a]$$

$$(15)$$

$$\frac{\partial\xi}{\partial T} = -\frac{(1 - \xi_A)}{2} a_M \sin[a_M(T - M_f) - b_M\sigma_a]$$

for the transformation from austenite to martensite.

3. Stress–strain relation under constant temperature

Under constant temperature, Equation 12 becomes an ordinal differential equation for σ and ε because $dT = 0$ and the relation between σ and ε can be obtained for given initial conditions. The material properties of the SMA are assumed and are listed in Table I.

Two types of fibre/resin matrices are considered. Their properties are listed in Table II.

The stress–strain relation for a pure SMA wire ($V_a = 1.0$, $V_m = 0$) under constant temperature $T_0 = 50$ °C is shown in Fig. 3. The initial martensite fraction, ξ_M, is assumed to be zero. In Fig. 3, the solid curve shows the solution when $D(\xi)$ is taken as the function of ξ according to Equation 3, while the dotted curve shows the solution when D is taken as a constant $D = 50$ GPa, the average of D_A and D_M. The difference between the two solutions under these two assumptions is obvious. Stage OA depicts the linear elastic deformation when the SMA is in the austenite phase. Stage AB shows the stress-induced martensitic transformation. In this stage, the strain increases significantly. At point B the martensitic transformation is finished. Stage BC is the elastic deformation of the SMA in the martensite phase.

Stress–strain relations of SMA-reinforced graphite/epoxy (curve I) and glass/epoxy (curve II) under constant temperature, $T_0 = 25\,°C$, are shown in Fig. 4. The volume fraction of SMA is 20% ($V_a = 0.2$, $V_m = 0.8$). The SMA wires are laid in the direction of the matrix fibres. Owing to the low fraction of SMA, the lamina property is dominated by the matrix material so that the stress–strain relation is nearly linear.

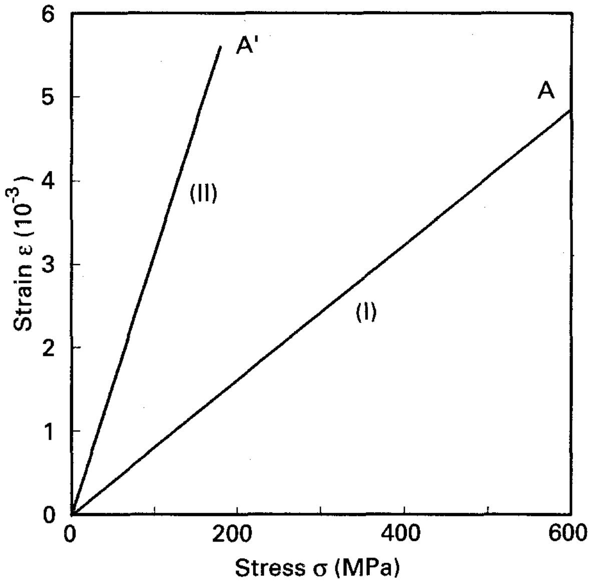

Figure 4 Lamina σ–ε relation ($T_0 = 25\,°C$) for (I) graphite/epoxy matrix, (II) glass/epoxy matrix.

It should be pointed out that slight non-linearity can be seen for both materials in the neighbourhood of point 0 when the deformation starts. When the martensitic transformation starts, however, the curves become almost linear. Within the range shown in Fig. 4, the martensite transformation is unfinished. The martensite fractions at point A (for graphite/epoxy matrix) and point A′ (for glass/epoxy matrix) are $\xi = 0.41$ and 0.476, respectively. Also, owing to the low fraction of the SMA, the solution with variable $D(\xi)$ nearly coincides with the solution of taking D as a constant (the average of D_A and D_M), so that the dotted curve is indistinguishable from the solid curve for both materials.

The stress–strain relation curve when the SMA wires (with volume fraction $V_a = 0.2$) are laid perpendicular to fibres of the graphite/epoxy matrix, is shown in Fig. 5. The solid curve and the dotted curve are the results of variable $D(\xi)$ and constant D, respectively. In this case, the influence of the modulus of SMA is increased because the modulus of the matrix in the direction of the SMA wires is lower than the previous cases. However, the solid curve is still indistinguishable from the dotted curve until the end of martensitic transformation at point A.

4. Free recovery and restrained recovery

The strain of a pure SMA wire with increasing temperature under external stress-free conditions ($\sigma = 0$, $d\sigma = 0$) is plotted in Fig. 6. The initial temperature $T_0 = 20\,°C$, and the initial martensite fraction $\xi_0 = 1$. The austenitic transformation of SMA starts when the wire is heated to $T = A_s = 38\,°C$ associated with volume contraction. In the present work, the contractive

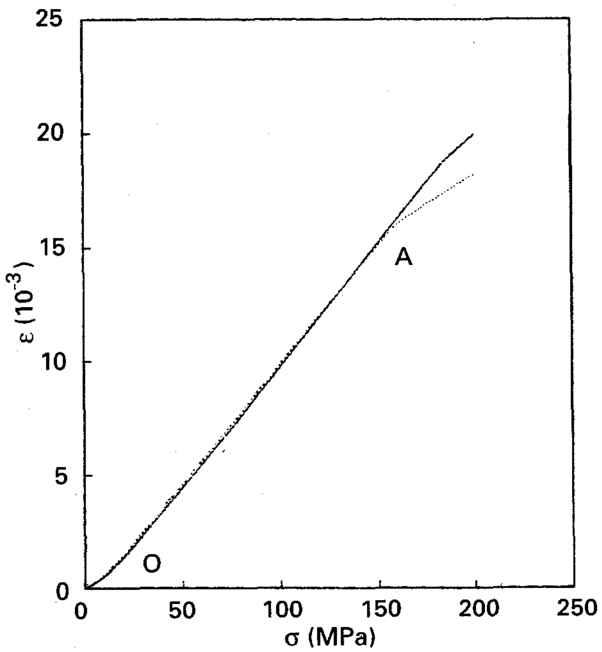

Figure 5 Lamina σ–ε relation when graphite fibres are perpendicular to the SMA wires.

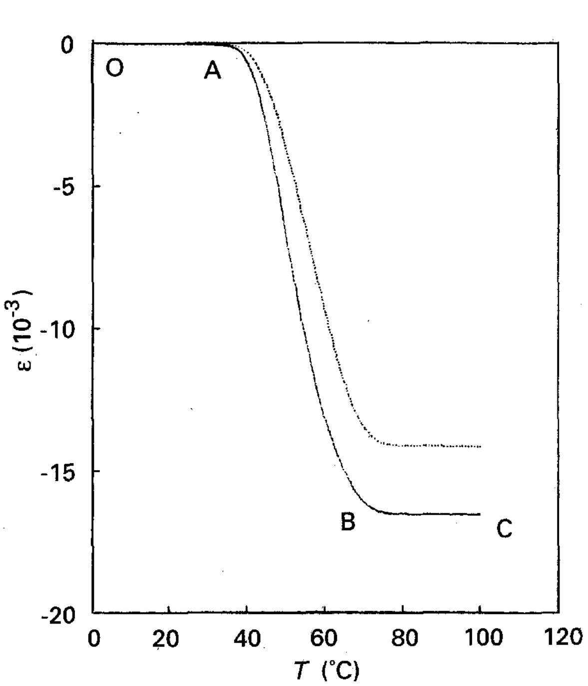

Figure 6 Free recovery of SMA wire.

deformation when heating with zero external stress ($\sigma = 0$) is referred to as free recovery. Because the thermo-elastic coefficient, θ, of the SMA considered is positive, it contracts upon heating. However, the volume change due to thermo-elastic deformation is much less than that caused by phase transformation. It is seen in Fig. 6, that the stage OA is the smallest thermo-elastic contraction. Stage AB is the largest contractive deformation mainly due to the transformation from martensite phase ($\xi = 1$) to austenite phase ($\xi = 0$). Stage BC is the thermo-elastic contraction after phase transformation. The dotted curve is the result when D is taken as a constant. The difference between the two curves is obvious.

The result of free recovery of SMA-reinforced glass/epoxy with SMA volume fraction $V_a = 0.2$ is shown in Fig. 7. The SMA wires are laid parallel to the fibres. It is seen that the lamina strain, ε, is positive before $T = 40\,°C$ (Stage OA) when the thermal expansion of the matrix is dominant. The austenitic transformation starts at point A and the contraction of SMA begins. The contractive strain of the lamina is small because the speed of contraction of SMA is slowed down by the action of the matrix thermal expansion. At the same time, high tensile stress, σ_a, is built up in the SMA wires. The speed of the austenitic transformation of SMA is also slowed down, owing to the high stress, σ_a. The stress of embedded SMA wires, σ_a, is plotted in Fig. 8 where the martensite volume fraction, ξ, at point B is 0.384.

Finally, the lamina stress, σ, and the SMA stress, σ_a, versus increasing temperature under the fully restrained condition ($\varepsilon = 0$) are plotted in Fig. 9. It should be noted that in the case of zero strain, $\varepsilon = 0$, the stress of the SMA, σ_a, is uncoupled with the stress in the matrix, σ_m. The lamina stress, σ, can be obtained from the equation

$$\sigma = V_a\sigma_a + V_m\sigma_m \qquad (16)$$

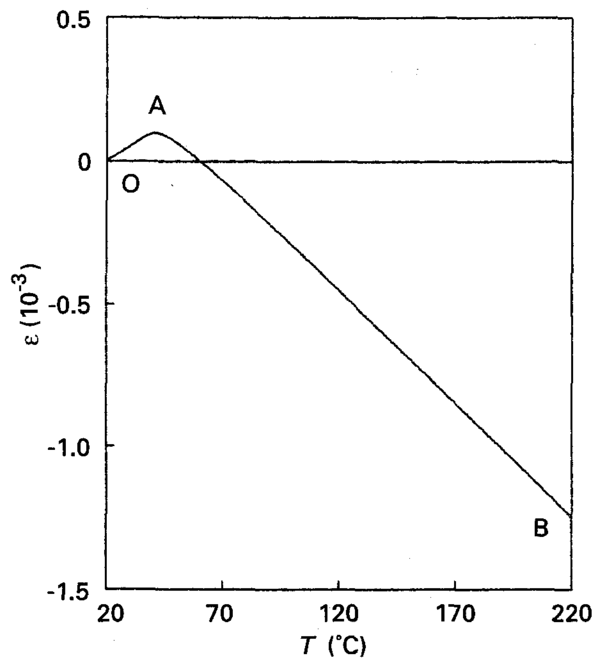

Figure 7 Free recovery of SMA/glass/epoxy lamina.

In fact, the fully restrained condition at the ends is not realistic. The case of the restrained condition at the ends can be simplified as a linear spring with elastic constant, k (see Fig. 10); the incremental stress–strain relation of the lamina can be written as

$$d\sigma = \frac{-kL}{A} d\varepsilon \qquad (17)$$

where L and A are the length and cross-sectional area of the lamina. By substituting Equation 17 into Equation 12, the relation of stress or strain versus temperature under elastic constraint conditions can be obtained.

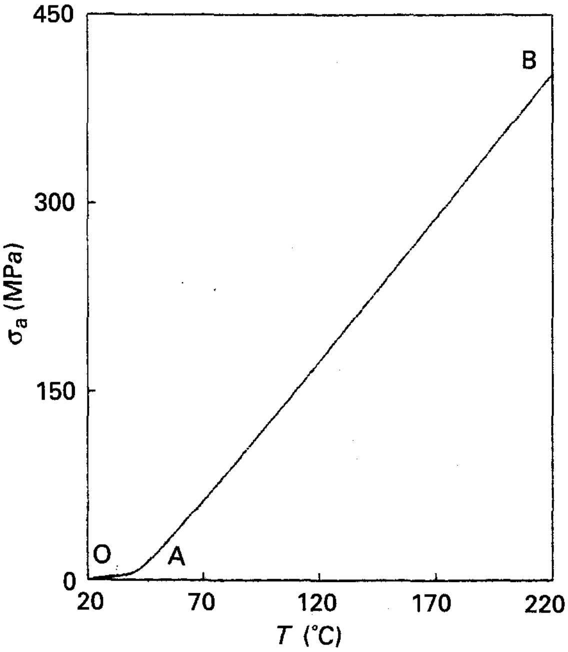

Figure 8 SMA stress, σ_a, in SMA/glass/epoxy lamina.

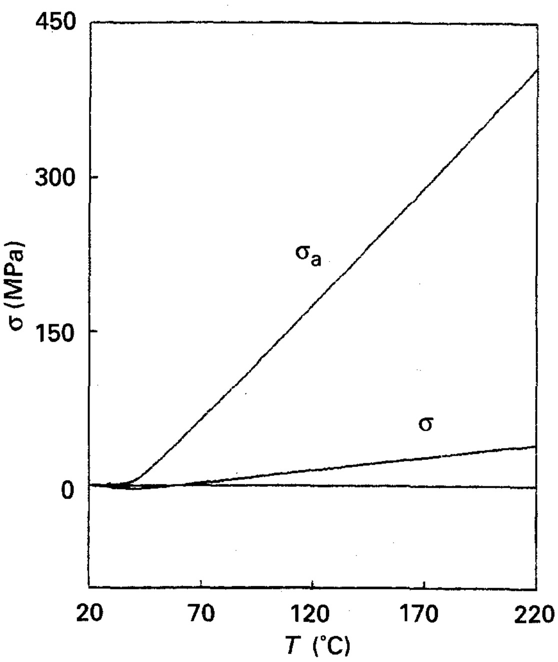

Figure 9 Restrained recovery of SMA/glass/epoxy lamina.

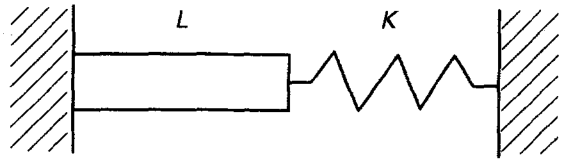

Figure 10 Recovery under elastic restraint.

5. Conclusion

The mechanical behaviour of shape-memory alloys caused by temperature-induced or stress-induced phase transformation is very complex. It is difficult to use a simple model or a single formula to describe all the processes caused by martensitic transformation or the reverse transformation. However, the prospect for applying SMA-reinforced composites, i.e. the smart structures, is bright. In the present study, the one-dimensional thermomechanical constitutive relation of SMA-reinforced composite lamina is derived and the dependence of the modulus of SMA on the phase composition is taken into account. The stress–strain relation under constant temperature, the free recovery and the restrained recovery of the SMA-reinforced composites are analysed. It is demonstrated that for a pure SMA wire, the difference of the results by taking $D(\xi)$ as variable or a constant is substantial. In the case of SMA wires embedded in a composite matrix with volume fraction $V_a = 0.2$, the influence on the laminar behaviour of taking D as a variable is insignificant, probably due to the low volume fraction of SMA. However, if the matrix modulus in the SMA wire direction is low, this effect cannot be neglected.

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